Subject: Leaving Certificate Maths

Teacher: Mr Murphy

Lesson 10: Coord Geo of the Line

10.1 Learning Intentions

After this week's lesson you will be able to;

- Calculate the distance between two points
- Calculate the midpoint of a line
- Establish the slope of a line
- Explain the relationship between perpendicular and parallel lines
- Generate the equation of a line
- -Establish the point of intersection between two lines
- Calculate the area of a triangle using coordinates
- Calculate the perpendicular distance between a point and a line
- Find the value of an angle between two lines

10.2 Specification

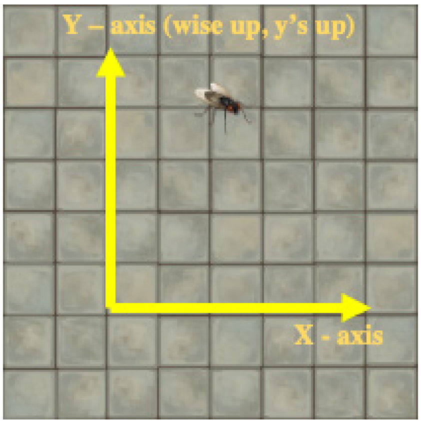
2.2 Co-ordinate geometry	– use slopes to show that two lines are • parallel • perpendicular – recognise the fact that the relationship $ax + by + c = 0$ is linear – solve problems involving slopes of lines – calculate the area of a triangle – recognise that $(x-h)^2 + (y-k)^2 = r^2$ represents the relationship between the x and y co-ordinates of points on a circle with centre (h, k) and radius r – solve problems involving a line and a circle with centre (0, 0)	– solve problems involving • the perpendicular distance from a point to a line • the angle between two lines – divide a line segment internally in a given ratio m: n – recognise that $x^2+y^2+2gx+2fy+c = 0$ represents the relationship between the x and y co-ordinates of points on a circle with centre $(-g,-f)$ and radius r where $r = \sqrt{(g^2+f^2-c)}$ – solve problems involving a line and a circle

10.3 Chief Examiner's Report

In many topics, including coordinate geometry and trigonometry, drawing sketches or diagram may aid candidate in understanding how to solve the problem.

10.4 Introduction to Coordinate Geometry

Co-ordinate geometry was first started by a French mathematician René Descartes (1596 – 1650). He was a Jesuit educated man, who was a friend of Pascal (another famous mathematician). He was never in good health but one day as he lay sick in bed he looked up at the ceiling, and this is what was there:

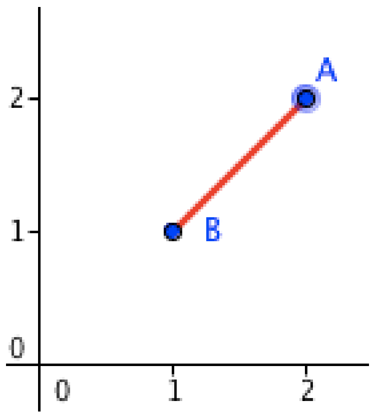


What he wanted was to be able to describe the exact position of the fly. So he decided that if I describe how far the fly is from a vertical line and a horizontal line that will give a precise location. Thus Co-ordinates in the way we know now was born.

In memory of René the (x,y) co-ordinate system is sometimes referred to as the Cartesian system.

10.5 Distance between two points

If we take two points A and B, we can use a particular formula to establish how far apart these two points are. This distance formula will always provide the straight-line distance between two.

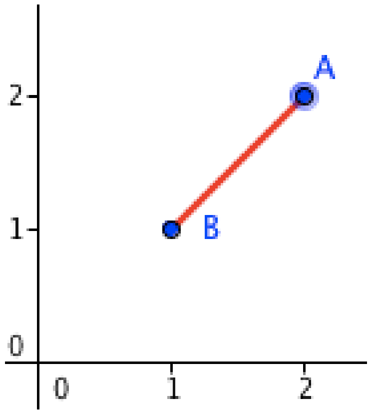


$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Sometimes you might not actually need the formula. On the axes below sketch two different situations where you could find the distance between two points **without** the formula.

10.6 Mid-Point of a line segment

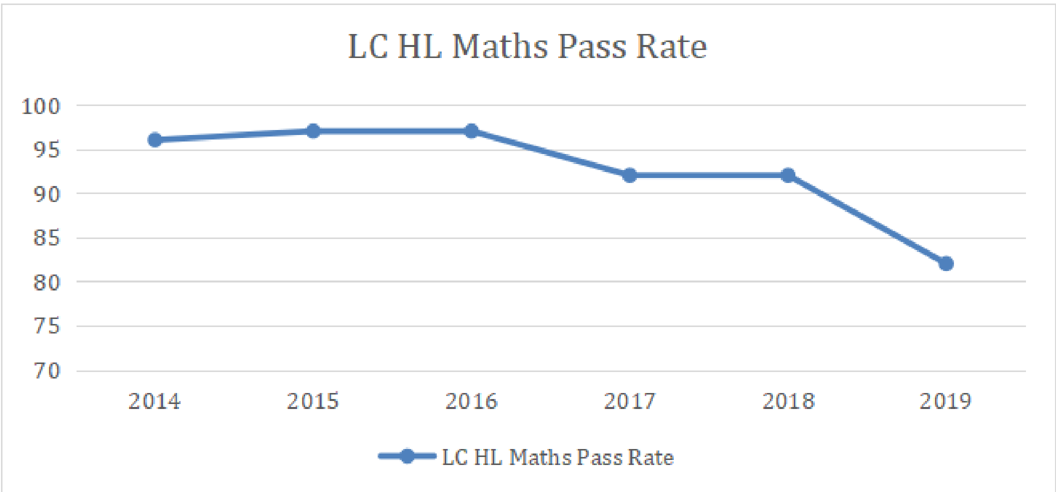
If we take the same two points A and B. Let's say we want to find the halfway point of these two, there is also a formula for that. This halfway point is often referred to as the mid-point of the line segment AB.



$$Mid - Pt. = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

10.7 Slope

The slope is probably the most widely used aspect of coordinate geometry and with that, one of the most important.



If we look at the graph on the left you can see a number of points where the line changes direction.

Label these points A-F.

Describe the line segments as either increasing or decreasing:

[AB] = [BC] =

[CD] = [DE] =

[EF] =

How did you judge if they were increasing or decreasing?

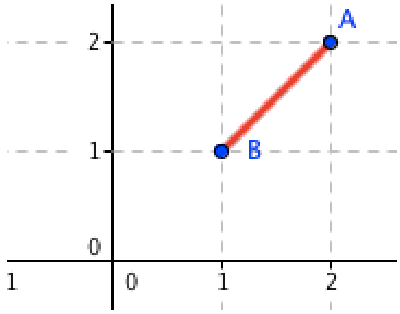
You based your answer on **slope**. It's easy to see if a line is sloping upwards (increasing) or downwards (decreasing) but what we need to do is be more exact about how fast a segment is increasing or decreasing. For this we need to calculate a value for slope.

There are two* methods of doing this;

1) Using a formula:

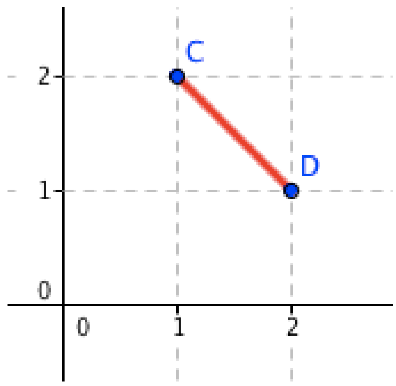
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

2) Rise over Run



We can also get the situation where the slope is decreasing:

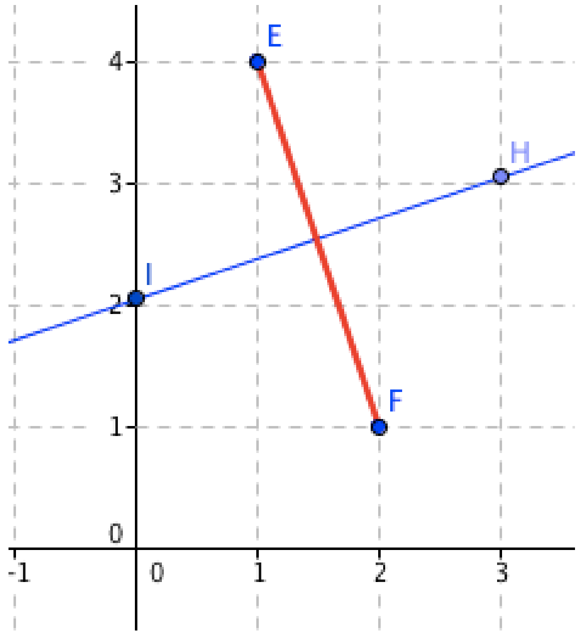
Calculate the slope of the line segment [CD] using the formula:



Compare the slopes of AB and CD, if a slope is increasing it will be **positive** and if a slope is decreasing it will be **negative**.

If using the rise over run method for the slope, as the line is dropping not rising we include a negative sign in the slope

Let's now look at a line and a line segment together:



Slope of [EF] =

Slope of [IH] =

How could you describe the relationship between these two lines?

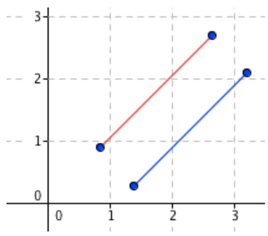
What about the relationship between their slopes?

If two lines are ______________ to one another then the product of their slopes is -1. As in if we multiply their slopes the answer will always be -1

NB This means that if we know the slope of one line and want the slope of a line that is perpendicular to it. We turn the slope value upside down and multiply it by -1.

For example, if the line L has a slope of $^{3}/_{4}$ then a line K, which is perpendicular to L has a slope of $^{-4}/_{3}$.

Now let's have a look at this pair of lines:



What can you say about their slopes?

These two lines have the **SAME** slope, these are known as parallel lines.

10.8 Equation of a Line

This is one of the most important aspects of co-ordinate geometry. The equation of a line tells us important information about the line.

The best way to have the equation of the line is in the below form:

$y = mx + c$

Sometimes however you can get the equation of a line expressed in a different way such as:

3x + 2y = 12

In this case it is a good idea to rearrange the equation to the y = mx + c format like this:

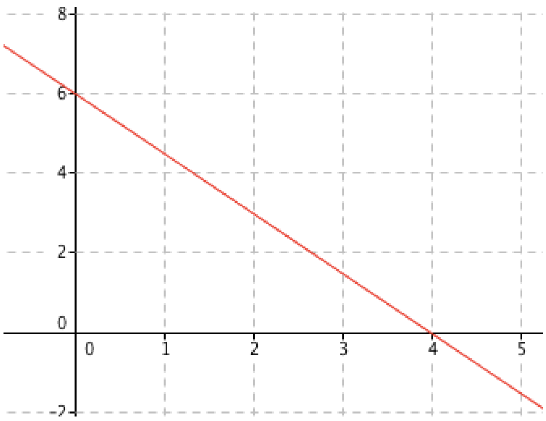
$$3x + 2y = 12$$

$$2y = 12 - 3x$$

$$2y = -3x + 12$$

$$y = -\frac{3}{2}x + 6$$

With the equation in this format we can see that the slope is -3/2 and the y-intercept is +6, let's see this as a graph:



You may also be asked to create the equation of a line. In this case you need one point that is on the line and also the slope of the line. To do this you use the below formula:

$$y - y_1 = m(x - x_1)$$

10.8 Equation of a Line

Another skill you will need is to verify if a point is on a line. For example; verify the point (3,2) is on the line K, $y = 2x - 4$.

$$y = 2x - 4$$
$$2 = 2(3) - 4$$
$$2 = 6 - 4$$
$$2 = 2$$
$$Q.E.D.$$

10.9 Points of Intersection

Find the point where two lines cross is quite a valuable tool and is relatively easy to do as it is something you have done in algebra.

to find the P.O.I. we need to use simultaneous equations.

Which we covered in the algebra notes. Recall these skills below by finding the P.O.I. between K and L.

$K: x + 2y - 4 = 0$

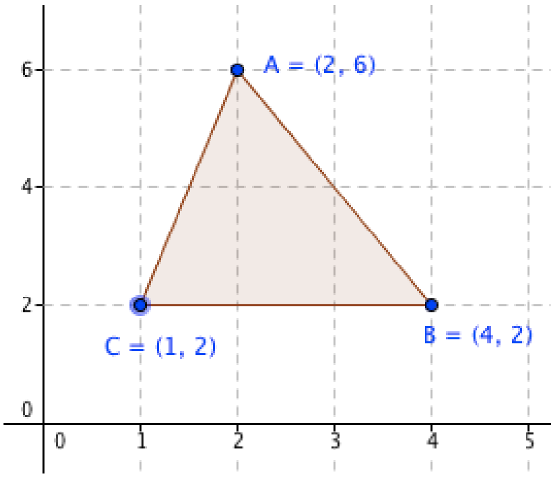
$L: x - y - 1 = 0$

10.10 Area of a Triangle

Coordinate Geometry can be used to find the area of a Triangle if we have the co-ordinates of its vertices (corners).

****However, one condition on this method working is that one of the vertices must have coordinates (0,0). If not, then using a translation we move it to (0,0).**

Take the triangle ABC below:



As we can see, none of the vertices are at (0,0) which is called the origin. So, we must now move one of them.

Pick whichever point would be the easiest to move, i.e. closest to the origin.

In this case it's C.

C = (1, 2) —> (0,0)   to do this we subtracted 1 from the x coordinate and subtracted 2 from the y coordinate.

REPEAT this for both point A and B, this is called a translation

B = (4, 2) —> (3, 0)

A = (2, 6) —> (1, 4)

Once we have carried out the translation we can now use the formula:

$$Area = \frac{1}{2}|x_1y_2 - x_2y_1|$$

$$Area = \frac{1}{2}|(3)(4) - (1)(0)|$$

$$Area = 6\ sq.units$$

10.11 Perpendicular Distance from a Point to a Line

This formula allows us to establish the shortest distance from a point onto a line. To do this we use the formula:

$$Perp.\ Distance = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

Again, notice the use of the absolute value symbol due to the idea of a negative distance being impossible.
This formula is used when we have a line of the form:

$$ax + by + c = 0$$

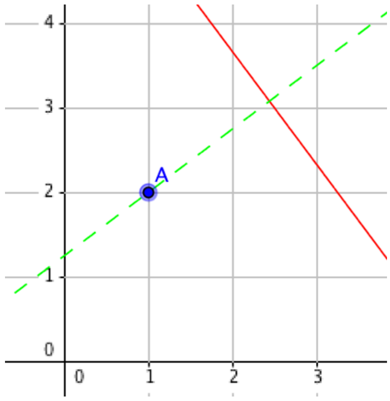
And a point of the from:

Point: (x_1, y_1)

From the video we have:

$4x + 3y - 19 = 0$

$A = (1,2)$



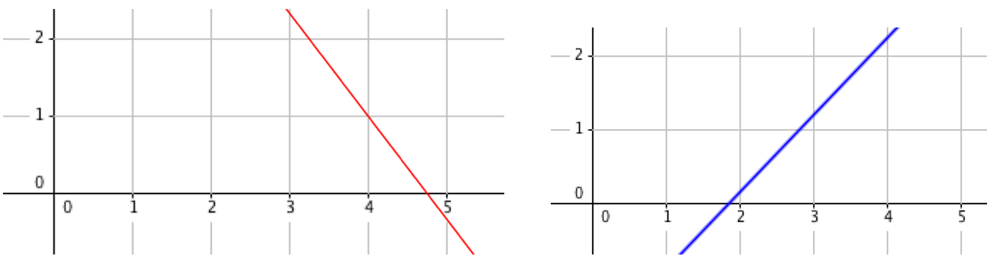
$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$d = \left|\frac{4(1) + 3(2) - 19}{\sqrt{(4)^2 + (3)^2}}\right|$$

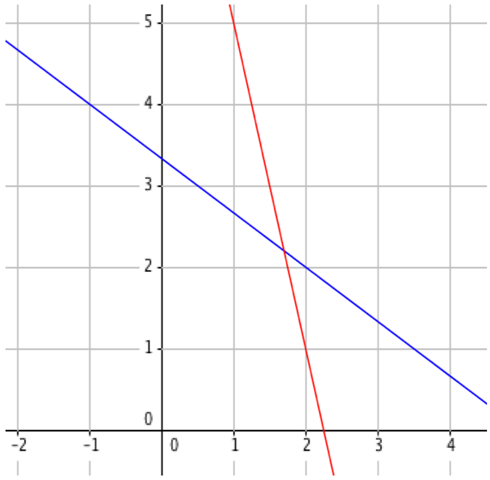
$$d = |1.8| = 1.8$$

10.12 Angle between two lines

Fill in the angle of inclination on both diagrams below:



Calculating this angle or the angle between two lines requires knowing the value of the slope of the lines



Calculate the slopes of these two lines first:

$m_1 =$

$m_2 =$

Now that we have the slopes, we can use the formula below:

$$tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

10.13 Recap of the Learning intentions

- Calculate the distance between two points
- Calculate the midpoint of a line
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10.14 Homework Task

i) The co-ordinates of two points are A(4, -1) and B(7, t). The line $l_1: 3x - 4y - 12 = 0$ is perpendicular to AB. Find the value of t.

ii) Find, in terms of k, the distance between the point P(10, k) and l_1.

iii) P(10, k) is on a bisector of the angles between the lines l_1 and $l_2: 5x + 12y - 20 = 0$. Find the possible values of k.

iv) If k > 0, find the distance from P to l_1.

10.15 Solutions to 9.14

$i)$ *Find all values of x for which* $\cos(2x) = -\frac{\sqrt{3}}{2}$, *where* $0° \leq x \leq 360°$

As the value of cos *is negative we are looking at the 2nd and 3rd quadrants.*

$\cos(2x) = -\frac{\sqrt{3}}{2}$

$2x = \cos\left(-\frac{\sqrt{3}}{2}\right)$

$Key\ angle = 30°$

2nd Quadrant:

$180° - Key\ angle = 150°$

$2x = 150 + 360n$

$x = 75 + 180n$

$x = 75 + 180(0) \qquad x = 75 + 180(1)$

$x = 75° \qquad x = 225°$

3rd Quadrant:

$180° + Key\ angle = 210°$

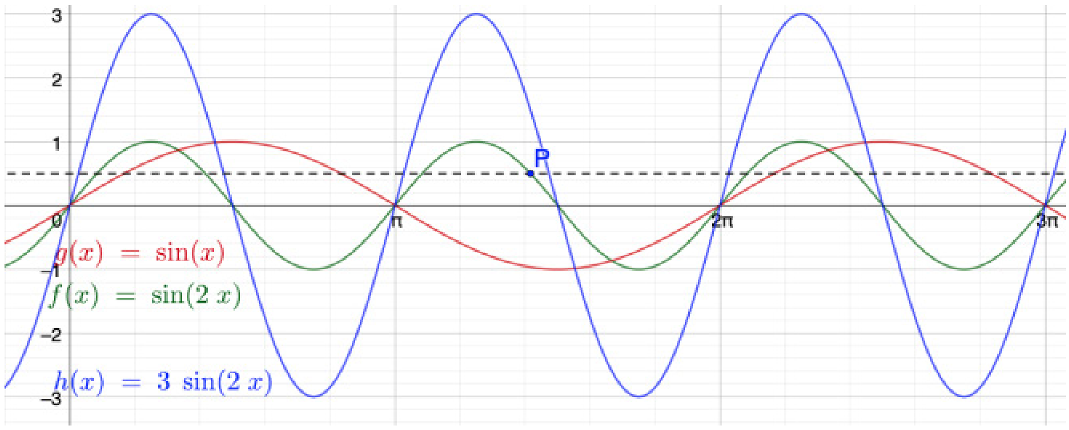
$2x = 210 + 360n$

$x = 105 + 180n$

$x = 105 + 180(0) \qquad x = 105 + 180(1)$

$x = 105° \qquad x = 285°$

$ii)$



Co-ordinates of P:
As P lies on both the black dashed line and the green function, we are looking for a P.O.I. between these functions.
As we can see there are many, but we only want this one.

$\sin(2x) = {}^{1}/_{2}$ now we have a equation just like in part one of the HW task.

$Key\ angle = {}^{\pi}/_{6}$ *using π because the $x-axis$ is in terms of π. So we are in the 1st and 2nd Quadrants*

$1st: 2x = {}^{\pi}/_{6} + 2n\pi \qquad 2nd: 2x = {}^{5\pi}/_{6} + 2n\pi$

$1st: 2x = {}^{\pi}/_{12} + n\pi \qquad 2nd: 2x = {}^{5\pi}/_{12} + n\pi$

So depending on the value of n we can have many possibilities for the x-coordinate of a P.O.I.

$\ldots {}^{\pi}/_{12}, {}^{5\pi}/_{12}, {}^{13\pi}/_{12}, {}^{17\pi}/_{12}, {}^{25\pi}/_{12}, {}^{29\pi}/_{12}, {}^{37\pi}/_{12} \ldots$

If we look at where P is we can see which one of these answers makes sense, ${}^{17\pi}/_{12}$.

P = $({}^{5\pi}/_{12}, {}^{1}/_{2})$